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Vibrational properties of an elastic continuum with dislocations and disclinations: a gauge approach

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Abstract. We study, in the framework of a gauge approach, the problem of small vibrations in an isotropic elastic continuum against the background of a static distortion due to a linear defect. In the approximation, which is linear in the dynamical displacements, the equations of motion for an isotropic elastic continuum with a topological defect are obtained. An analysis of the vibrational spectrum is presented for elastic materials with a screw dislocation, a wedge disclination and a disclination monopole. In the case of a screw dislocation the known results of classic defect theory are reproduced. For a wedge disclination, with a small non-integer Frank index, only minor changes of the spectrum are found to appear. It is shown that the situation is changed remarkably for topologically stable disclinations. In this case we deal with another topological sector, and the character of the vibrations differs essentially from that for topologically unstable defects.

1. Introduction

The dynamical properties of elastic media can be modified in the presence of linear defects such as dislocations and disclinations. As is known (see, for example, [1]), dislocations can affect lattice vibrations in two ways. The first possibility comes from large deformations in the core region of a defect. As a consequence, the elastic moduli are substantially perturbed near the dislocation line. This can result in the localization of vibrational modes which are split from the edge of the continuum spectrum of bulk acoustic waves [2]. The second possibility is due to the fact [3, 4] that elastic stresses caused by a dislocation can also be a reason for the localization of small harmonic vibrations against the background of the deformed lattice. At the same time, the dynamical properties of disclinated media are less well known. The reason is that only defects with a small Frank index can be considered in the framework of the standard theory, which is based essentially on the linear connection between stresses and strains.

In this paper we study the long-wave elastic vibrations in the isotropic defect media within the framework of the gauge theory of dislocations and disclinations formulated first in [5]. This theory is essentially non-linear in origin. It should be pointed out that there currently exist several powerful mathematical methods for the description of elastic continua with topological defects. One of them includes the pure topological analysis of defects in solid continua [6, 7]. Probably the most natural and elegant method was developed in [8, 9], where the differential geometric representation of dislocation theory was established. Unfortunately, the application of the methods of non-Riemannian geometry in defect theory meets with considerable mathematical difficulties, particularly beyond the perturbation theory analysis (see, for example, [10]).

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An alternative approach is the gauge theory of defects in continuum media [5, 11]. It turns out that, like the known gauge approach first introduced by Utiyama in gravitational theory (see, for example, a discussion on the applicability of gauge constructions in gravitational theory in [12]), the gauge model is quite suitable for the description of defect media. Moreover, it was shown recently [13] that some predictions of the differential geometrical approach in defect theory are recovered within the gauge model, thus indicating the possible similarity of these methods. On the other hand, the gauge approach was shown to be successful in the study of topologically stable rotational defects with integer Frank indices. In particular, the exact solutions of the completely non-linear problems of a disclination vortex and disclination monopole were found in [14, 15]. Since the dynamics of elastic media containing topological defects is of interest, it is reasonable to study this problem within the gauge approach.

The paper is structured as follows. In section 2 we construct the general scheme and obtain the equation describing small vibrations in the presence of a linear defect. The case of topologically unstable defects will be considered in section 3. To illustrate the method, we reproduce the known vibrational spectrum of a body with a straight screw dislocation. We also study radial acoustic waves in a thin elastic disk with a fractional disclination vortex (the straight wedge disclination). The character of the vibrations in the presence of topologically stable disclinations (a disclination vortex and a disclination monopole) is studied in section 4. Section 5 is devoted to concluding comments.

2. The general scheme

Elastic vibrations in isotropic defect media can be studied by considering small vibrations against the background of a statically deformed lattice. In the framework of the gauge approach, this can be realized by including small dynamical displacements in the state vector $\chi^i(x, t)$. Namely, we can write

$$\chi^i(x, t) = \tilde{\chi}^i(x) + u^i(x, t) \qquad (1)$$

where $\tilde{\chi}^i(x)$ denotes the state vector in the presence of a static defect, and $u^i(x, t)$ are the small dynamical displacements. It is clear that such a representation will generate the form of all other tensors presented in the gauge model (see [5]). For instance, the distortion tensor takes the following form:

$$B^i_a = \tilde{B}^i_a + \partial_a u^i \qquad (2)$$

where $\tilde{B}^i_a = \partial_a \tilde{\chi}^i + H^i_a$, $H^i_a = \epsilon^i_{\alpha k} W^\alpha_a \chi^k + \phi^i_a$ in three space dimensions ($d = 3$) and $H^i_a = \epsilon^i_j W_a \chi^j + \phi^i_a$ in the planar case ($d = 2$). Here completely antisymmetric tensors $\epsilon^i_{\alpha k}$ and ϵ^i_j are the generating matrices of the gauge groups $SO(3)$ and $SO(2)$, and W^α_a and ϕ^i_a are the compensating gauge fields associated with the disclination and dislocation fields, respectively. Summation over repeated indices is assumed, $a = \{A, 4\}$, $\partial_A = (\partial/\partial x^A)$, $\partial_4 = (\partial/\partial_t)$. Greek indices and capital letters take their values from sets $\{1, 2, 3\}$ for $d = 3$ and $\{1, 2\}$ for $d = 2$, respectively. The strain tensor is determined to be

$$E_{AB} = \tilde{E}_{AB} + \tilde{B}^i_B \partial_A u^i + \tilde{B}^i_A \partial_B u^i \qquad (3)$$

where $\tilde{E}_{AB} = \tilde{B}^i_A \tilde{B}^i_B - \delta_{AB}$. Hereafter we restrict consideration to an approximation that is linear in the dynamical displacements. To avoid cumbersome expressions we will sometimes omit the right order of the top and bottom indices, which can be easily restored by using the appropriate δ symbols. The stress tensor takes the following form:

$$\sigma_i^A = \tilde{\sigma}_i^A + U_i^A \tag{4}$$

where

$$U_i^A = \tfrac{1}{2}\tilde{B}^i_C[\lambda\delta^{AC}\mathrm{Sp}(\tilde{B}^j_B\partial_D u^j + \tilde{B}^j_D\partial_B u^j) + 2\mu(\tilde{B}^j_C\partial_A u^j + +\tilde{B}^j_A\partial_C u^j)] \\ + \tfrac{1}{2}(\partial_C u^i)[\lambda\delta^{AC}\mathrm{Sp}\tilde{E}_{AB} + 2\mu\tilde{E}_{AC}] \tag{5}$$

and $\mathrm{Sp}\tilde{E}_{AB} = \tilde{E}_{AA} = \tilde{E}_{xx} + \tilde{E}_{yy} + \tilde{E}_{zz}$, with λ and μ being the Lamé constants. As in the classical theory of elasticity, we will keep in (4) only the terms that are linear in the strain tensor. In this case, the static stress tensor is determined to be

$$\tilde{\sigma}_i^A = \tfrac{1}{2}\tilde{B}^i_C(\lambda\delta^{AC}\mathrm{Sp}\tilde{E}_{AB} + 2\mu\tilde{E}_{AC}). \tag{6}$$

Since we are interested only in the vibrational properties of defect media we can restrict our consideration to the equation of balance of the linear momentum presented in [5, 16]. With (1)–(6) taken into account, it can be written as

$$\partial_t p_i - \partial_A\sigma_i^A = P_i \tag{7}$$

where the momentum $p_i = \rho_0\delta_{ij}(\partial_t\chi^j + H_4^j)$, ρ_0 is the mass density in the reference configuration, and

$$P_i = \epsilon^j_{\alpha i}(W_4^\alpha p_j - W_A^\alpha\sigma_j^A + F^\alpha_{ab}R_j^{ab}) \tag{8}$$

in three space dimensions (the gauge group $G = T(3) \triangleright SO(3)$) [5], and

$$P_i = \epsilon^j_i(W_3 p_j - W_A\sigma_j^A + F_{ab}R_j^{ab}) \tag{9}$$

in two space dimensions with the gauge group $G = T(2) \triangleright SO(2)$ [16]. We will not clarify here the explicit form of the tensors F and R (for details see [5, 16]) since, in fact, we will consider in this paper only problems where either dislocations or disclinations are present in the elastic continuum. In both cases the last term does not appear in the right-hand sides of (8) and (9). We assume also that gauge fields due to defects are static ones. This correlates with the problem since we study the dynamics of an elastic continuum against the background of the static defect. In this case, (7) can be simplified. Namely, one gets $p^i = \rho_0\partial_t u^i$, $P_i = \tilde{P}_i + P_i^*$ where $\tilde{P}_i = -\epsilon^j_{\alpha i}W_A^\alpha\tilde{\sigma}_j^A$, $P_i^* = -\epsilon^j_{\alpha i}W_A^\alpha U_j^A$ for $d = 3$, and $\tilde{P}_i = -\epsilon^j_i W_A\tilde{\sigma}_j^A$, $P_i^* = -\epsilon^j_i W_A U_j^A$ for $d = 2$, so that finally

$$\rho_0\partial_t^2 u_i - \partial_A U_i^A = P_i^* \tag{10}$$

where the static equations $-\partial_A\tilde{\sigma}_i^A = \tilde{P}_i$ are assumed to be fulfilled. Let us emphasize that (10) is the most general form of the dynamical field equation in an approximation that is linear in displacements u^i. In order to study (10) we need the explicit form of the distortion tensor $\tilde{B}^i_A$ in the static case. Fortunately, there are some exact solutions of the static defect problems [5, 14, 15, 17] which allow us to study the dynamical problem in detail.

3. Topologically unstable defects

A topological analysis shows [6] that defect states of the medium may be classified by topologically stable classes (the homotopy classes). The trivial class contains the reference configuration and topologically unstable defects. Defects of this sort can be transformed into the homogeneous state by a local alteration of the order parameter. On the other hand, defects from the first and other homotopical classes are topologically stable ones. In this section we will consider linear defects that belong to the trivial class. The static state vector for elastic media with such defects can be written as $\tilde{\chi}^i(x^C) = \delta^i_B x^B + \tilde{u}^i(x^C)$ where $\tilde{u}^i(x)$ are the static displacements caused by a defect. In this case, (2) takes the form

$$\tilde{B}^i_A = \delta^i_A + T^i_A \tag{11}$$

where $T^i_A = \partial_A \tilde{u}^i + H^i_A$ determines the contribution to the distortion tensor associated with a defect. When $T^i_A = 0$, we get the distortion tensor of the reference (homogeneous) configuration, and thus classic elasticity theory is recovered [5]. It is of importance that T^i_A turns out to be proportional to either one of the components of the Burgers vector b or the Frank index ν. As is known, in the linear theory of defects the values of b and ν are assumed to be small, so that perturbation techniques may be applied. In the following we will keep only terms that are linear in b or ν. The tensor U^A_i, linearized in such a manner, takes the following form:

$$U^A_i = \lambda\delta^A_i \partial_B u^B + \mu(\partial_A u^i + \partial_i u^A) + G^A_i \tag{12}$$

where

$$G^A_i = \lambda\delta^A_i T^j_B \partial_B u^j + \mu\delta^C_i (T^j_C \partial_A u^j + T^A_j \partial_C u^j) + \lambda T^A_i \partial_B u^B + \mu T^C_i (\partial_A u^C + \partial_C u^A) + \lambda(\mathrm{Sp} T^j_B)\partial_A u^i + \mu(T^A_C + T^C_A)\partial_C u^i . \tag{13}$$

Finally, the equation describing small vibrations of isotropic elastic media with topologically unstable defects is written as

$$\rho_0 \partial^2_t u_i - \mu\Delta u_i - (\lambda + \mu)\partial_i \partial_A u^A = J_i \tag{14}$$

where $\Delta = \partial^2_x + \partial^2_y + \partial^2_z$, and

$$J_i = \partial_A G^A_i + P^*_i . \tag{15}$$

Notice that when there are no defects, the right-hand side of (14) turns out to be zero and we recover the equation of motion for isotropic elastic media which is well known in the classical theory of elasticity [1]. As is also known, the solutions of (14) in the defect-free case are longitudinal and transverse waves with dispersion laws $\omega = k_l v_l$ and $\omega = k_t v_t$, respectively. Here $v_l = [(\lambda + 2\mu)/\rho_0]^{1/2}$ and $v_t = (\mu/\rho_0)^{1/2}$ are the corresponding sound velocities.

3.1. Dislocations

In this subsection we consider disclination-free materials. In this case, there is no breaking of the homogeneity of the action of the rotation group, so that one has to put gauge fields W_A^α equal to zero in all formulae. In particular, the form of J_i in (14) will be essentially simplified. To illustrate the method presented in the previous section, let us consider here the simple example of a straight screw dislocation oriented in the zth direction. This problem was studied earlier in [3,4] within a classical approach. It is interesting to compare the classical results with those obtained in the framework of the gauge approach. In the linear approximation, a solution for this defect was found to be [11,13]

$$\phi_3^1 = \phi_1^3 = -\left(\frac{b}{2\pi}\right)\frac{y}{r^2} \qquad \phi_3^2 = \phi_2^3 = \left(\frac{b}{2\pi}\right)\frac{x}{r^2} \tag{16}$$

where other components as well the static displacements vanish, $r^2 = x^2 + y^2$, and b is the zth component of the Burgers vector. As is known [6], there are problems with applying the topological method to media with broken translational invariance in the uniform state. The reason is that, in general, the order parameter cannot be properly defined in such systems. In crystals, however, line defects are characterized by Bravais lattice vectors and the very existence of these vectors allows one to determine the topologically stable classes. On the other hand, in an elastic continuum there are no formal restrictions on the value of the Burgers vector and thus we consider the solution (16) as belonging to the trivial topological class. As is seen, for (16) one has $\mathrm{Sp}\phi_A^i = 0$ and $\partial_A\phi_A^i = 0$. Therefore, one gets

$$\begin{aligned} J_i = {} & 2\mu\phi_A^i\Delta u^A + (\lambda+\mu)\phi_A^j\partial_A\partial_i u^j + (\lambda+\mu)\phi_B^i\partial_A\partial_B u^A + 2\mu\phi_B^A\partial_A\partial_B u^i \\ & + 2\mu(\partial_A\phi_B^i)(\partial_A u^B) + \mu(\partial_A\phi_B^i)(\partial_B u^A) + \lambda(\partial_i\phi_B^j)(\partial_B u^j). \end{aligned} \tag{17}$$

Let us consider an elastic wave $u^z(r,\theta,z,t)$ propagating along the dislocation line. In this case J_i takes the form

$$J_i = A(\phi_1^3\partial_x\partial_z u^z + \phi_2^3\partial_y\partial_z u^z) \tag{18}$$

where $A = 6\mu + 2\lambda$. One can easily check that this expression is just the same as that found within the standard approach [3,4]. The solution of (14) with (18) is well known, namely [3,4]

$$u^z(r,\theta,z,t) = \chi(r)\exp[\mathrm{i}k(z + b\theta/2\pi) + \mathrm{i}m\theta - \mathrm{i}\omega t]. \tag{19}$$

For $m = 0$ the spectrum was found to contain localized levels with a dispersion law $v_l^2k^2 - \omega_{\mathrm{loc}}^2 \sim \exp(-\Lambda/b|k|)$ where Λ is a characteristic parameter (see details given in [3,4]). It was noted that this modification of the spectrum must be taken into account in the study of the low-temperature thermodynamic properties of defect crystals.

3.2. Disclinations

In this subsection we will study the case when only disclinations are present in the isotropic elastic continuum. It is clear that now terms with ϕ_A^i have to be omitted in (14). We are able to analyse (14) only knowing the explicit form of static fields W_A and displacements $\tilde{u}^i$. Recently, in the framework of a linearized gauge model, the exact solution for a fractional disclination vortex has been obtained [17]. It was shown that the stress and strain fields associated with the disclination vortex are just the same as those for a straight wedge disclination with a non-integer (but small) Frank index in classical theory. When the disclination is oriented along the z axis the problem becomes, in fact, a two-dimensional one in the plane normal to the disclination line. In this case, the following solution was found [17]:

$$W_A = -\nu\epsilon_C^A \frac{x^C}{r^2} \tag{20}$$

where ν is the Frank index and

$$\tilde{u}^i = x^i(C_1 \ln r + C_2). \tag{21}$$

Here $r^2 = x^2 + y^2$, $C_1 = -\nu/(L+2)$, $L = \lambda/\mu$ and C_2 is an arbitrary constant which can be fixed by using the appropriate boundary conditions. In general, stress fields caused by a disclination turn out to diverge both at $r \to 0$ and $r \to \infty$. As is known, this difficulty may be avoided by considering a single defect in a sample of finite size. The classical problem of this sort is a straight wedge disclination in a cylinder or in a thin disk. Let us consider small vibrations in the plane normal to the disclination line. In cylindrical coordinates (r, θ), (14) takes the following form for acoustic radial waves:

$$\rho_0 \partial_t^2 u_r - (\lambda + 2\mu)\Delta_R u_r = J_r. \tag{22}$$

Here $\Delta_R u = \partial_r^2 u + (1/r)\partial_r u - (1/r^2)u$, and we have supposed that $u_r = u = u(r,t)$ does not depend on θ because of isotropy. Then, J_r is written as

$$J_r = A(r)\partial_r^2 u + \frac{1}{r}B(r)\partial_r u - \frac{1}{r^2}C(r)u \tag{23}$$

where $A(r) = h(r) + (3\lambda + 6\mu)C_1 - \lambda\nu$, $B(r) = h(r) + (7\lambda + 12\mu)C_1$, and $C(r) = h(r) - \lambda C_1 - (4\lambda + 8\mu)\nu$ where $h(r) = (4\lambda + 6\mu)(C_1 \ln r + C_2)$.

Let us consider harmonic vibrations, choosing $u(r,t) = u(r)\cos(\omega t + \gamma)$. For a cylinder of radius R the constant C_2 was determined to be [17] $C_2 = (\nu/2) + [\nu/(L+2)]\ln R$. Therefore, we get an equation for $u(r)$ in the form

$$\partial_r^2 u(r)[1 - \nu(f(r) - K_1)] + \frac{1}{r}\partial_r u(r)[1 - \nu(f(r) - K_2)] + \frac{1}{r^2}u(r)[k^2 r^2 - (1 - \nu(f(r) + K_3))] = 0. \tag{24}$$

Here $f(r) = 2(2L+3)\ln(r/R)$, $K_1 = L/(L+2)$, $K_2 = 2(L^2-3)/(L+2)^2$, $K_3 = 2(L+1)(L-5)/(L+2)^2$ and $k^2 = \omega^2/v_l^2$.

When $\nu = 0$, (24) describes the radial long-wave vibrations of a thin defect-free disk and takes the form

$$\partial_r^2 u_0(r) + \frac{1}{r}\partial_r u_0(r) + \left(k_0^2 - \frac{1}{r^2}\right)u_0(r) = 0. \tag{25}$$

This is the well known Bessel equation, with a solution $u_0(r) = J_1(k_0 r)$ where J_1 is the corresponding Bessel function. Thus, in the defect-free case, the radial waves are $u_0(r,t) = AJ_1(\omega r/v_l)\cos(\omega t + \gamma)$ where A is a constant. The vibrational modes can be found from the boundary condition $J_1(\omega R/v_l) = 0$. They are $\omega = \omega_0^n = v_l\alpha_n/R$, where α_n is the nth zero of the Bessel function $J_1(x)$. The general solution can be written as

$$u_0(r,t) = \sum_n A_n J_1(\alpha_n r/R)\cos(\omega_0^n t + \gamma_n). \tag{26}$$

These are harmonic vibrations with frequency ω_0^n and an r-dependent amplitude $A_n J_1(\alpha_n r/R)$. To analyse (24) in the presence of a defect, let us take into account the fact that the Frank index ν is assumed to have a small value, so that one can consider ν as a small dimensionless parameter and conventional perturbation technique can be used (for details see [18]). Namely, the solution of (24) can be written as $u(r) = u_0(r) + \nu u_1(r) + \cdots$, and $\omega = \omega_0 + \nu\omega_1 + \cdots$. In this case, the zeroth-order equation is given by (25) whereas in the first-order approximation one gets

$$\partial_r^2 u_1(r) + \frac{1}{r}\partial_r u_1(r) + \frac{\omega_0^2}{v_l^2}u_1(r) = F(\omega_1, \omega_0, u_0, \ldots). \tag{27}$$

We will not specify the form of the right-hand side of (27) since we do not plan to study (27) in detail here. In some respects this problem is close to the problem of a round membrane considered in [18]. Even without solving (27) one can conclude that only minor changes of the vibrational spectrum will appear. It should be emphasized, however, that we do not consider here the perturbation of the elastic moduli in the core region of the defect which can also result in the localized long-wave modes. In addition, within the continuum approximation used, we cannot discuss the role of modes with short wavelengths in the vibrational spectrum.

4. Topologically stable defects

As was noted in [5], the linear elasticity approximation based upon expansion in the scaling parameter of the gauge group is effective because of the assumption that the reference configuration is defect free. A topological analysis shows [6], however, that an elastic continuum can contain defects of a different kind. These defects belong to the non-trivial topologically stable homotopy classes, i.e. they cannot be eliminated by a homogeneous deformation of the order parameter. In this case, the reference configuration turns out to be in another topological sector and, to study the topologically stable defects, we have to discard the assumption that the reference configuration is defect free [5]. As a consequence, the distortion tensor cannot be presented in a simple form (11). Hence one has to turn back to (10) with U_i^A determined in (5). As previously, we need the explicit form of the distortion tensor. Notice that in the framework of the gauge approach it has been possible to obtain two exact solutions for the topologically stable disclinations, such as a disclination monopole [14] and an integer disclination vortex [15]. This provides the detailed analysis of the vibrational problem in disclinated materials.

4.1. Disclination vortex

An exact solution for a static disclination vortex with an integer Frank index was found to be [15, 16]

$$W_A(x^C) = -n\epsilon_C^A\frac{x^C}{r^2} \qquad \chi^1(x) = F(r)\cos(n\theta) \qquad \chi^2(x) = F(r)\sin(n\theta) \tag{28}$$

where n can have any integer value $(1, 2, 3, \ldots)$. We will consider here the case $n = 1$. For this solution the distortion tensor takes the form

$$\tilde{B}^i_A = \frac{x^i x_A}{r^2} g(r). \tag{29}$$

Here the function $g(r) = \partial_r F(r)$ was found to be $g(r) = N_0 \tilde{g}(r)$, where

$$\tilde{g}(r) = \begin{cases} \tilde{g}_1(r) = \cosh[\frac{1}{3}\cosh^{-1}(r_0/r)] & r \leqslant r_0 \\ \tilde{g}_2(r) = -\cos[\frac{1}{3}\cos^{-1}(r_0/r) + \frac{2}{3}\pi l] & r \geqslant r_0 \end{cases} \tag{30}$$

with r_0 a characteristic parameter that defines the core region of the defect, $N_0^2 = 8(\lambda + \mu)/3(\lambda + 2\mu)$, and $l = 0, 1, 2$. In the following we restrict ourselves to the case $l = 0$. Notice that (29) differs drastically from (11). Indeed, as is known, in the gauge model [5] the metric tensor can be determined as $g_{AB} = B^i_A B^i_B$. One can see that for (11) the metrics of the reference configuration only slightly differ from the Euclidean one (δ_{AB}) whereas this is not the case for (29). The strain tensor is determined to be $\tilde{E}_{AB} = \tilde{B}^i_A \tilde{B}^i_B - \delta_{AB} = (x_A x_B / r^2) g^2(r) - \delta_{AB}$. It should be noted that both strains and stresses do not diverge as $r \to \infty$. To study the vibrational states of the disclinated material, let us consider (10). Using (28) and (29) we obtain that

$$U^A_i = (\lambda + \mu) g^2(r) \frac{x^i x^A x^j x^B}{r^4} (\partial_B u^j) + \mu g^2(r) \left(\frac{x^i x^j}{r^2} (\partial_A u^j) + \frac{x^A x^C}{r^2} (\partial_C u^i) \right) + \tfrac{1}{2}\lambda (g^2(r) - 2)(\partial_A u^i) - \mu (\partial_A u^i) \tag{31}$$

and

$$P^*_i = \epsilon^j_i \epsilon^A_C \frac{x^C}{r^2} \left(\mu g^2(r) \frac{x^j x^B}{r^2} (\partial_A u^B) + \tfrac{1}{2}\lambda g^2(r)(\partial_A u^j) - (\lambda + \mu)(\partial_A u^j) \right). \tag{32}$$

For the sake of convenience, let us study (10) in cylindrical coordinates. We will consider harmonic acoustic radial waves: $u^r = u(r, t) = u(r)\cos(\omega t + \gamma)$. In this case, the final equation for $u(r)$ is found to be

$$\partial_r^2 u(r) + \frac{1}{r}[1 + r\partial_r \ln f(r)] \partial_r u(r) + \frac{\rho_0 \omega^2}{f(r)} u(r) = 0 \tag{33}$$

where $f(r) = (\lambda + \mu)(4\tilde{g}^2(r) - 1)$. It is well known (see, for example, [19]) that an equation of the type $u'' + P(x)u' + Q(x)u = 0$ can be put into the form $z'' + I(x)z = 0$ by the substitution $u(x) = z(x)\exp[-(1/2)\int P(x)\mathrm{d}x]$. It is easy to check that in our case

$$\exp\left(-\frac{1}{2}\int P(r)\mathrm{d}r\right) = \tfrac{1}{2}[rf(r)]^{-1/2}. \tag{34}$$

Thus, instead of (33) we obtain

$$\partial_r^2 z(r) + I(r) z(r) = 0 \tag{35}$$

where

$$I(r) = \frac{\rho_0 \omega^2}{f(r)} - \tfrac{1}{4}F^2(r) - \tfrac{1}{2}\partial_r F(r) \tag{36}$$

and $F(r) = \partial_r \ln[rf(r)]$. Let us analyse (35). Although a general solution of such equations does not exist, important information can nevertheless be obtained from the asymptotic solutions. In view of (30) one obtains that, at large distances from the disclination line, $I(r) \to \omega^2/v_s^2$ where $v_s^2 = 2(\lambda+\mu)/\rho_0$. It is important to note that the velocity v_s differs from the sound velocity v_l in the defect-free case. Thus, vibrations take the form

$$u(r,t)|_{r\to\infty} \to \frac{A}{\sqrt{r}} \cos\left(\frac{\omega}{v_s} r + \phi\right) \cos(\omega t + \gamma) \tag{37}$$

where A is a constant. At the same time, for $r \to 0$, $I(r)$ tends to infinity as r^{-2}, and the asymptotic solutions for $u(r)$ take the form $u(r)|_{r\to 0} \sim r^{2/3}$ and $u(r)|_{r\to 0} \to u_0$ where u_0 is a constant. An interesting question arises as to the character of vibrations in the core region. Within the classical approach it is assumed that phonons do not penetrate the core region [1,4]. Since in our case $g(r)$ is determined over a wide space interval including the core region we can study this problem in detail. The analysis shows that, for $r \to r_0$, $I(r) \to \rho_0\omega^2/3(\lambda+\mu) + (C/r_0^2)$ where $C > 0$. Thus, in our case, vibrations will penetrate the core region of the disclination. It is important to note that for the long-wave (small-ω) vibrations which are of interest here, the first term in (36) becomes negligible in the core region. Hence $u(r)$ becomes independent of ω and v_s in this region. It differs essentially from the defect-free case where all points of the elastic continuum oscillate harmonically with the same frequency ω_k^n (see (26)). It should be noted, however, that the results concerning the core region become incorrect when r_0 approaches the lattice constant. In this case the continuum description itself ceases to be true.

4.2. Disclination monopole

The exact solution for a disclination monopole was obtained first in [14]. In contrast to the linear defects considered so far in this paper, a disclination monopole is a point-like defect. In many respects it looks like a 'hedgehog' known in liquid crystals (see, for example, [20]). Let us study briefly the vibrational properties of an elastic continuum with a disclination monopole. The solution for this defect takes the form

$$W_A^\alpha(x^B) = \epsilon_{AB}^\alpha \frac{x^B}{r^2} \qquad \chi^i(x^A) = F(r)\frac{x^i}{r}. \tag{38}$$

Here $r^2 = x^A x_A = x^2 + y^2 + z^2$. The Frank index for this solution is $n = 1$. As is seen, this solution possesses spherical symmetry. The distortion tensor has the same form as (29) where, however, $N_0^2 = 4(3\lambda + 2\mu)/3(\lambda + 2\mu)$ and

$$\tilde{g}(r) = \begin{cases} \tilde{g}_1(r) = \cosh[\frac{1}{3}\cosh^{-1}(r_0/r)^2] & r \leqslant r_0 \\ \tilde{g}_2(r) = -\cos[\frac{1}{3}\cos^{-1}(r_0/r)^2 + \frac{2}{3}\pi l] & r \geqslant r_0. \end{cases} \tag{39}$$

One can easily check that the expression for U_i^A is almost the same as (31) with only the replacement $\mathrm{Sp}\tilde{E}_{AB} = g^2(r) - 3$ taken into account. In this case, the last but one term in (31) takes the form $\frac{1}{2}\lambda(g^2(r) - 3)$. Accordingly, the last term in (32) is modified as $(3\lambda/2 + \mu)\partial_A u^j$. Finally, (33) is rewritten as

$$\partial_r^2 u(r) + \frac{1}{r}[2 + r\partial_r \ln f(r)]\partial_r u(r) + \frac{\rho_0\omega^2}{f(r)} u(r) = 0 \tag{40}$$

where $f(r) = (3\lambda/2 + \mu)(4\tilde{g}^2(r) - 1)$.

In the classical theory of elasticity the radial vibrations of an elastic sphere with a dispersion law $\omega = kv_l$, where admissible frequencies are defined by the boundary condition $\sigma_r^r(R) = 0$, are well known (see, for example, [20]). Suppose we place the disclination monopole at the centre of a sphere. In this case, we have to analyse (40). This can be done in the same manner as in the previous subsection. Namely, one can write (40) in the form (35) by the substitution $z(r) = u(r)r\sqrt{f(r)}$. At large distances we get

$$u(r,t)|_{r\to\infty} \to \frac{A}{r}\cos\left(\frac{\omega}{v_s}r + \phi\right)\cos(\omega t + \gamma) \tag{41}$$

where $v_s^2 = (3\lambda + 2\mu)/\rho_0$. This asymptotic solution is the same as in the defect-free case, but v_s again exceeds the sound velocity v_l. As $r \to 0$, the r-dependent solution for $u(r)$ tends to zero more rapidly than that for the vortex, namely $u(r)|_{r\to 0} \sim r^{4/3}$. All the conclusions made at the end of the previous subsection are valid here as well.

5. Conclusion

In this paper we have presented a new approach, based on the gauge theory of defects, which allows us to describe the vibrational properties of defect elastic continua. We have derived the general equations describing small vibrations in defect media. As an illustration of the method we reproduced the known results for materials with straight screw dislocations. We have considered the long-wave elastic vibrations in isotropic materials with straight wedge disclinations and a disclination monopole. The analysis shows that the character of vibrations in the presence of a topologically unstable disclination only slightly differs from that in defect-free materials. When topologically stable defects are present, the character of the vibrations is found to be essentially altered, especially in the core region. Depending on the geometry of the problem, at long distances from the defect line there are ordinary radial vibrations (planar or spherical) but the sound velocity is found to exceed that in a defect-free elastic body. In the vicinity of the disclination line the amplitude $u(r)$ becomes independent of the frequency and sound velocity and has power-like asymptotics.

As is known, disclinations can play an important role in disordered materials, like amorphous bodies, glasses, liquid crystals, polymers etc. In particular, recent progress in the theoretical description of liquids and metallic glasses has been inspired by the new point of view of their structure. An attractive model was proposed by Kléman and Sadoc [21] who showed that dense random packing can be carried out by the mapping into flat space of tetrahedra which tile a space with an appropriately chosen curvature. This mapping leads to various kinds of defects, including disclinations. This idea was developed by Nelson, who considered disclinations as the fundamental defects in metallic glasses [22]. An interesting idea has also been proposed by Rivier who considered topologically stable line defects in glasses [23]. It would be very interesting therefore to apply our approach to the description of spectral properties in these materials. To this end, we have to extend the gauge model in order to take into account the dense packing of defects. A detailed analysis of this important problem requires further investigation.

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